Spin-current induced electric field

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We theoretically predict that a pure steady state spin-current without charge-current can induce an electric field. A formula for the induced electric field is derived and we investigate its characteristics. Conversely, a moving spin is affected by an external electric field and we present a formula for the interaction energy.

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In a traditional electric circuit the number of spin-up and spin-down electrons are the same, and both kinds of electrons move in the same direction under an external electric field. The total spin-current $I_s = \sigma(I_\uparrow - I_\downarrow)$ is therefore zero, and only the charge-current $I_{ec} = e(I_\uparrow + I_\downarrow)$ is relevant. When a system includes ferromagnetic materials or under an external magnetic field, electron spins can be polarized so that the total spin of the system is non-zero. Then, the corresponding charge-current is polarized, *i.e.* current due to spin-up electrons, $I_\uparrow$, is not equal to the spin-down current $I_\downarrow$, although both kinds of electrons move in the same direction, as schematically shown in Fig.1b. This gives a non-zero total spin-current. Spin-polarized charge-current has been the subject of extensive investigations for last two decades.[1,2] Recently, a very interesting extreme case of a finite spin-current without charge-current has been investigated by several groups[3–5]. Such a situation comes about when spin-up electrons move to one direction while an equal number of spin-down electrons move to the opposite direction, as schematically shown in Fig.1a. Then the total charge-current is identically zero and only a net spin-current exists. This is just the opposite situation of the traditional charge-current without any spin.

By Ampere's law, a charge-current induces a magnetic field in the space around it. In this paper, we ask and answer the following question: can a pure steady state spin-current without charge-current induce an electric field? The problem can be viewed in the following way. Associated with the electron spin σ, there is a magnetic moment $g\mu_B\sigma$, where μ_B is the Bohr magneton and g is a constant. Therefore when there is a spin-current I_s, there is a corresponding magnetic moment current $I_m = g\mu_B I_s$. In the rest of the paper, we theoretically prove that a magnetic moment current can induce an electric field. We further prove that an external electric field can also act on a spin-current.

To start, we recall that a static classical magnetic moment $\mathbf{m}$ produces a magnetic field $\mathbf{B}$. Consider a classical magnetic moment $\mathbf{m}$ due to a tiny charge-current ring, see Fig.1c. The charge-current is I_{ec} and radius of the ring is δ. The magnetic field $\mathbf{B}$ of this charge-current ring is easily obtained by the Biot-Savart law. Then, let $\delta \to 0^+$ and $I_{ec} \to \infty$ but keep $\mathbf{m} = \pi\delta^2 I_{ec}\hat{n}_m$ as a constant ($\hat{n}_m$ is the unit vector of the magnetic moment), the magnetic field $\mathbf{B}$ due to magnetic moment $\mathbf{m}$, at space point $\mathbf{R}$, can be written as:

$$\mathbf{B} = -\nabla \frac{\mu_0 \mathbf{m} \bullet \mathbf{R}}{4\pi R^3} \quad . \tag{1}$$

Another method for obtaining the same magnetic field is by using the mathematical construction of equivalent magnetic "charge"[6]. In this method, we imagine the magnetic moment $\mathbf{m}$ being consisted of a positive and a negative magnetic "charge" $\pm q_{mc}$ situated very close to each other with a distance δ (see Fig.1d). When $\delta \to 0^+$ and $q_{mc} \to \infty$, we hold $\mathbf{m} = q_{mc}\delta\hat{n}_m$ as a constant. Each magnetic "charge" q_{mc} produces a magnetic field $\frac{\mu_0 q_{mc}\mathbf{R}}{4\pi R^3}$. The field $\mathbf{B}$ induced by magnetic moment $\mathbf{m}$ can then be obtained by adding contributions of the two magnetic "charges" $\pm q_{mc}$. Of course, we again obtain Eq.(1). Note that the language of magnetic "charge" is only a mathematical construction convenient for our derivations[6], and no magnetic monopole is hinted whatsoever.

After reviewing the magnetic field of a *static* magnetic moment, in the rest of the paper we consider magnetic moments in motion. In the first example we consider the simplest case of a classical infinitely long one-dimensional (1D) lattice of chargeless magnetic moment, with the whole lattice moving with speed $\mathbf{v}$ (see Fig.1e). This gives a pure steady state magnetic moment current. Since nothing is changing with time, a very surprising result, as we now prove, is that this magnetic moment current induces an *electric* field. The induced field for this situation can be calculated exactly by a simple Lorentz transform, therefore providing a benchmark result for our more general results to be discussed later. Let ρ_m to be the linear density of magnetic moment for the lattice and we will use two methods to solve the electromagnetic field of the spin-current.

Method One. We first solve the total *magnetic* field of a *static* 1D magnetic moment lattice by integrating Eq.(1) over the lattice. This is easy to do and we call the result $\mathbf{B}_{static}$: $\mathbf{B}_{static} = -\nabla \frac{\mu_0 \rho_m \hat{n}_m \bullet \mathbf{R}_\perp}{2\pi R_\perp^2}$, where $\mathbf{R}_\perp = \mathbf{R} - (\mathbf{R} \bullet \hat{l})$. Here $\hat{l}$ is unit vector along the lattice. Then we make a relativistic transformation: the electromagnetic field of the moving magnetic moment lattice can be obtained straightforwardly by the Lorentz trans-

form of $\mathbf{B}_{static}$. The results are

$$\mathbf{B} = \gamma \mathbf{B}_{static} \ , \tag{2}$$

$$\mathbf{E} = -\gamma \mathbf{v} \times \mathbf{B}_{static} \ , \tag{3}$$

where $\gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$. Clearly, we have an induced electric field $\mathbf{E}$ and this field is related to $\mathbf{v}$. We note that although the results are unambiguously obtained, we have not identified the physical origin of the resulting electromagnetic field, *i.e.* this method does not tell us whether the field is induced by the magnetic moment or by the magnetic moment current. For this reason, we analyze the same problem again from a second method.

Method Two. Here we use the equivalent magnetic "charge" method discussed above. This means removing the current density of the ring at the equation $\nabla\times\mathbf{B} = \mu_0\mathbf{J}_{ec}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$, and adding the imaginary magnetic charge at the equation $\nabla\bullet\mathbf{B}=0$, *i.e.* this equation changes to $\nabla\bullet\mathbf{B}=\mu_0\rho_{mc}$, where ρ_{mc} is the volume density of magnetic "charge". We emphasize again that this practice is only a mathematical trick to solve our problem. When our magnetic moment moves, the original Maxwell equation in which the magnetic moment $\mathbf{m}$ is described by a tiny charge-current ring, satisfies relativistic covariance. Clearly, Maxwell equations after the equivalent magnetic charge transformation must also satisfy relativistic covariance. This covariance can be achieved, as shown in standard textbook[6], by changing the Maxwell equation to $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$ to $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} - \mu_0\mathbf{J}_{mc}$, where $\mathbf{J}_{mc}$ is the magnetic "charge" current.[6] The last equation means that a moving magnetic "charge" can produce an electric field. The electric field $\mathbf{E}$ produced by a volume (linear) element of magnetic "charge" current, $\mathbf{J}_{mc}dV$ ($I_{mc}d\mathbf{l}$), is simply:

$$\mathbf{E} = -\frac{\mu_0\mathbf{J_{mc}}dV\times\mathbf{R}}{4\pi R^3} = -\frac{\mu_0 I_{mc}d\mathbf{l}\times\mathbf{R}}{4\pi R^3} \ . \tag{4}$$

Now we are ready to solve the electromagnetic field of our 1D moving magnetic moment lattice because it is equivalent to two lines of positive/negative moving magnetic "charges": they are easily obtained by integrating Eqs.(1) and (4) respectively. The same final results of Eqs.(2,3) are obtained. The present derivation allows us to conclude that the magnetic field $\mathbf{B}$ is induced by magnetic moment and the electric field $\mathbf{E}$ is induced by the magnetic moment current. Fig.1f shows electric field lines and magnetic field lines at the y-z plane, here the infinitely long magnetic moment lattice is along the x-axis and $\hat{n}_m$ is along the +z-direction.

If there exists another infinitely long magnetic moment lattice with opposite magnetic moment direction ($-\hat{n}_m$) and opposite moving direction ($-\mathbf{v}$, shown in Fig.1a), then the net magnetic moment is canceled exactly and only a net magnetic moment current exists. In this case, it is easy to confirm that the magnetic field $\mathbf{B}$ due to each lattice adds up to zero identically, while the electric field $\mathbf{E}$ reinforce each other so that the total electric field of the composite system is doubled. Hence we conclude that this finite electric field must originate from the magnetic moment current, and it cannot be due any other effects.

In the example above, we have clearly shown that a moving classical 1D magnetic moment can induce an electric field $\mathbf{E}$. In the following we investigate the question: can moving electron magnetic moment (*i.e.* spin) induce an electric field? We also extend the above 1D model to general situation. Before proving this is indeed the case, we emphasize the fact that since a magnetic moment (or a spin) is itself a vector unlike charge which is a scalar, the magnetic moment current density cannot be described only by a single vector $\mathbf{J}_m dV$ (or $I_m d\mathbf{l}$). In order to completely describe a magnetic moment current density, we have to use a set of two vectors ($\hat{n}_m$, $\mathbf{J}_m dV$) or ($\hat{n}_m$, $I_m d\mathbf{l}$), in which $\mathbf{J}_m$ expresses the strength and direction of the flow of magnetic moment current, while $\hat{n}_m$ expresses the polarization of the magnetic moment itself. This is different from the familiar charge current. Note that for two magnetic moment current such as that of Fig.1a, if only their $\mathbf{J}_m$ are the same and their $\hat{n}_m$ are different, they are two different magnetic moment currents and their induce electric fields are also different (see below).

In the following, we apply the equivalent magnetic "charge" method to deduce a general result beyond 1D for the quantum object of electron spin-current. Here, the spin or magnetic moment $\mathbf{m}$ of an electron at space point $\mathbf{r}$ is equivalent to a positive magnetic charge $\frac{m}{\delta}$ at $\mathbf{r}+\frac{\delta}{2}\hat{n}_m$ and a negative magnetic charge $-\frac{m}{\delta}$ at $\mathbf{r}-\frac{\delta}{2}\hat{n}_m$. The spin-current ($\hat{n}_m$, $\mathbf{J}_m dV$) at the space $\mathbf{r}$ is equivalent to two magnetic "charge" currents: one is $\frac{\mathbf{J}_m}{\delta}dV$ at $\mathbf{r}+\frac{\delta}{2}\hat{n}_m$ and the other $-\frac{\mathbf{J}_m}{\delta}dV$ at $\mathbf{r}-\frac{\delta}{2}\hat{n}_m$, where $\delta\to 0^+$. We make the very reasonable fundamental assumption that any electromagnetic field induced by moving electron spins, if exists, must satisfy relativistic covariance. From this assumption, the Maxwell equations for the magnetic "charge" and its current are:

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} - \mu_0\mathbf{J}_{mc}, \tag{5}$$

$$\nabla\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \mu_0\mathbf{J}_{ec}, \tag{6}$$

$$\nabla\bullet\mathbf{E} = \rho_{ec}/\epsilon_0, \tag{7}$$

$$\nabla\bullet\mathbf{B} = \mu_o\rho_{mc}, \tag{8}$$

where ρ_{mc} and ρ_{ec} are the volume density of the magnetic and electric charge, $\mathbf{J}_{mc}$ and $\mathbf{J}_{ec}$ are their current density. In contrast, in the original Maxwell equation, the source of field are electric charge and its current: the field of a magnetic moment is calculated by turning this moment into an infinitesimal charge-current loop as we have done above. Here, we use the magnetic "charge" description and its associated current to express the spin of particles and the spin-current. We emphasize two points: (i) Eqs.(5-8) are superiorer for our problem of calculating fields of electron spin-current because they do not require us to turn electron spins into little charge-current loops.

No one knows how to do the latter, in fact, because the inner structure of an electron is not known. Hence, while the original Maxwell equations do not directly describe fields of electron spin and the spin-current, Eqs.(5-8) can describe them and this description is very reasonable if we only investigate fields outside of an electron, *e.g.* for $R > 10^{-5}$Å. (ii) Eqs.(5–8) do not represent an attempt of rewriting Maxwell equation. They *are* the Maxwell equation when we use equilent magnetic "charge" and demanding relativistic covariance.

From the Eqs.(5-8), the electric field induced by an infinitesimal element of magnetic "charge" current $\mathbf{J}_{mc}dV$ is obtained from Eq.(4). Then the total electric field $\mathbf{E}$ of the element of magnetic moment current ($\hat{n}_m, \mathbf{J}_m dV$) can be calculated by adding the two contributions of the two magnetic "charge" currents: $\frac{\mathbf{J}_m}{\delta}dV$ at $\frac{\delta}{2}\hat{n}_m$ and $-\frac{\mathbf{J}_m}{\delta}dV$ at $-\frac{\delta}{2}\hat{n}_m$ ($\delta \to 0^+$). We obtain

$$\mathbf{E} = \int \frac{\mu_0}{4\pi}\mathbf{J}_m dV \times \frac{1}{R^3}\left[\hat{n}_m - \frac{3\mathbf{R}\left(\mathbf{R} \bullet \hat{n}_m\right)}{R^2}\right] . \quad (9)$$

This is one of the main results of this paper. Eq.(9) clearly shows that the magnetic moment current ($\hat{n}_m$, $\mathbf{J}_m dV$) (*i.e.* spin-current (σ, $\mathbf{J}_s dV$)= ($\hat{n}_m$, $\frac{\mathbf{J}_m}{g\mu_B}dV$)) indeed can produce an electric field. This formula can be thought as the " Biot-Savart law" for spin-current induced electric field. We emphasize that in the derivation of Eq.(9), the only assumption made was that the electromagnetic field of the moving spin satisfies relativistic covariance. As a check, applying Eq.(9) to the 1D lattice exactly solved above, it is straightforward to perform the integration and obtain Eq.(3).

Some further discussion of our results are in order. An electron has its charge and magnetic moment (spin): charge produces electric field, charge-current produces magnetic field, spin produces magnetic field, and we have just shown that a steady state spin-current produces an electric field! (i) For the case of a spin-current without charge-current shown in Fig.1a, the total net charge is zero for our neutral system; the total charge-current is zero; and the total magnetic moment is also zero. The only non-zero quantity is the total spin-current (magnetic moment current). Our results predict even for this situation, an electric field is induced by the presence of spin-current. (ii) For the spin-polarized charge-current shown in Fig.1b, which have been extensively investigated recently[1,2], a charge-current, total magnetic moment, and a spin-current may all exist. In this case, the charge-current and magnetic moment produce magnetic field, and the spin-current produces electric field. (iii) For a closed-loop circuit in which a steady state spin-current flows, one can prove that the induced electric field $\mathbf{E}$ has the property $\oint_c \mathbf{E} \bullet d\mathbf{l} = 0$, where C is an arbitrary close contour not cutting the spin-current. This is true even when the spin-current threads the contour C: very different from the Ampere's law of magnetic field induced by a charge-current.

Fig.2 shows electric field lines of a spin-current element ($\hat{n}_m$, $\mathbf{J}_m dV$). The spin-current element is located at origin, $\mathbf{J}_m$ points to $+\mathbf{x}$ direction, and $\hat{n}_m$ is in the x-z plane. The angle between $\mathbf{J}_m$ and $\hat{n}_m$ is θ. Because the induced electric field $\mathbf{E}$ must be perpendicular to $\mathbf{J}_m$ (*i.e.* to $\mathbf{x}$ axis), we plot the field lines in the y-z plane at $x = -1$, 0, and $+1$. (i) For $\theta = \pi/2$, $\hat{n}_m \perp \mathbf{J}_m$. At $x = 0$, the field line configuration is similar (although not exactly the same) to that produced by an electric dipole at $-\mathbf{y}$ direction (Fig.2a). At $x = \pm 1$, the field lines have a mirror symmetry between upper and lower half y-z plane (Fig.2b). (ii) For $\theta = 0$, $\hat{n}_m \parallel \mathbf{J}_m$. At $x = 0$, $\mathbf{E} = 0$ for any y and z. At $x = \pm 1$, the field lines are concentric circles (Fig.2c and d). The center of the circles is at $y = z = 0$ where $\mathbf{E} = 0$. (iii) For $\theta = \pi/3$, the fields are shown in Fig.2(e,f) for $x = \pm 1$. In fact, this $\mathbf{E}$ can be decomposed into a summation of two terms corresponding to the fields of $\theta = \pi/2$ and $\theta = 0$. At $x = 0$, the field lines are similar to that shown in Fig.2a. It is worth to mention that from Fig.2, it is clearly shown that $\oint_c \mathbf{E} \bullet d\mathbf{l} \neq 0$. Notice that this is not contradictory to characteristic (iii) of the last paragraph, because there the electric field is produced by a steady closed-loop spin-current.

So far we have demonstrated that a spin-current can indeed induce an electric field. On the other hand, does external electric field have any effect on a spin-current? Consider a lab frame Σ' where there is an electromagnetic field ($\mathbf{E}', \mathbf{B}'$), and a *static* magnetic moment $\mathbf{m}'$. There is a potential energy $-\mathbf{m}' \bullet \mathbf{B}'$ but $\mathbf{m}'$ does not couple to $\mathbf{E}'$. Using the language of magnetic "charge" discussed above, we can in effect consider that a force $q_{mc}\mathbf{B}'$ is acting on the magnetic "charge" q_{mc}. Inside a new frame Σ moving with speed $-\mathbf{v}$ respect to the lab frame Σ', the magnetic moment $\mathbf{m}$ (or the magnetic charge q_{mc}) move with speed $+\mathbf{v}$. A Lorentz transform of the the four-momentum $p_\mu = (\mathbf{p}, \frac{iW}{c}) = (p_1, p_2, p_3, \frac{iW}{c})$ from frame Σ' to Σ gives the force $\mathbf{F}$ on the moving magnetic "charge" q_{mc} by the electromagnetic field $\mathbf{E}, \mathbf{B}$: $\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d\mathbf{p}}{d\tau}\frac{d\tau}{dt} = \left(\frac{dp_1'}{d\tau}, \gamma^{-1}\frac{dp_2'}{d\tau}, \gamma^{-1}\frac{dp_3'}{d\tau}\right) = q_{mc}(B_1', \gamma^{-1}B_2', \gamma^{-1}B_3') = q_{mc}\left(\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right)$ where τ is the proper time, t is the time, the quantity with a prime (without prime) is in frame Σ' (Σ), and the direction of p_1 is the same as velocity $\mathbf{v}$. Hence, a moving magnetic moment $\mathbf{m}$ (or spin σ) in an external electromagnetic field $\mathbf{E}, \mathbf{B}$ feels a torque: $\mathbf{m} \times \left(\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right)$, and the associated potential energy is:[7]

$$-\mathbf{m} \bullet \left(\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right) = -g\mu_B\sigma \bullet \left(\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right) . \quad (10)$$

Clearly, the term $-\mathbf{m}\bullet\mathbf{B}$ describes the action of magnetic field on $\mathbf{m}$ which is well known. There is a new term $\mathbf{m} \bullet \left(\frac{\mathbf{v}}{c^2} \times \mathbf{E}\right)$, and it obviously expresses the action of electric field $\mathbf{E}$ on the *moving* magnetic moment. We therefore conclude that when a particle with spin σ is moving inside an electric field $\mathbf{E}$, the spin prefers to orient to the direction of $-\mathbf{v} \times \mathbf{E}$.

At last, it should be mentioned some previous works have investigated effects of moving magnetic dipole $\mathbf{m}$ in early days of special relativity.[8] The main finding was that a moving magnetic dipole induces an electric dipole $\mathbf{P}_e = \frac{\mathbf{v}}{c^2} \times \mathbf{m}$. Notice the our present work is different. What we predicted is that a *steady state* spin-current can induce an electric field $\mathbf{E}$ (see Eq.(9)). This gives a new and fundamental source of electric field. In constract, a single moving moment does not give a steady state spin-current. Moreover, the results are different. For example, when $\mathbf{v} \parallel \mathbf{m}$, the induced dipole $\mathbf{p}_e = 0$ so that no electric field is induced and any torque $\mathbf{p}_e \times \mathbf{E}$ from external electric field vanishes. Our result on spin-current induced field, on the other hand, gives a non-zero $\mathbf{E}$ when $\mathbf{v} \parallel \mathbf{m}$.

So far we have found that a spin current can induce an electric field $\mathbf{E}$; and conversely, an external electric field puts a moment of force on a spin-current. The magnitudes of these effects can be estimated. Consider a spin current $(\hat{n}_m, \mathbf{J}_m)$ flowing in an infinitely long wire with crossection area of 2mm×2mm. Let $\hat{n}_m \perp \mathbf{J}_m$, take electron density is $10^{29}/m^3$ and a drift velocity $10^{-2}m/s$, then the spin-current induced electric field is equivalent to that of a potential difference $\sim 12\mu$V at distances -1.1mm and 1.1mm on either side of the wire. This electric potential is indeed very small, but is definitely nonzero and should be measurable using present technologies.

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[6] D. J. Griffiths, *introduction to electrodynamics*, P308 (Prentice hall, Press 1989).
[7] Notice where the electromagnetic field $\mathbf{E}$ and $\mathbf{B}$ are nonuniform, the motive magnetic moment $\mathbf{m}$ except have the moment of force $\mathbf{m} \times \left(\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right)$, it too has the net force $(\mathbf{m} \bullet \nabla)\left(\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right)$.
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FIG. 1. Schematic plots for the spin current with zero charge current (a), the spin-polarized current (b), the magnetic moment of a small current ring (c), the two equivalent magnetic charges of the magnetic moment (d), and the straight infinite long magnetic moment line (e). f shows the electric (solid) and magnetic (dotted) line of force for the motive magnetic moment line.

FIG. 2. Schematic plots of electric field lines for a spin-current element with $\theta = \pi/2, \pi/3$ and 0.

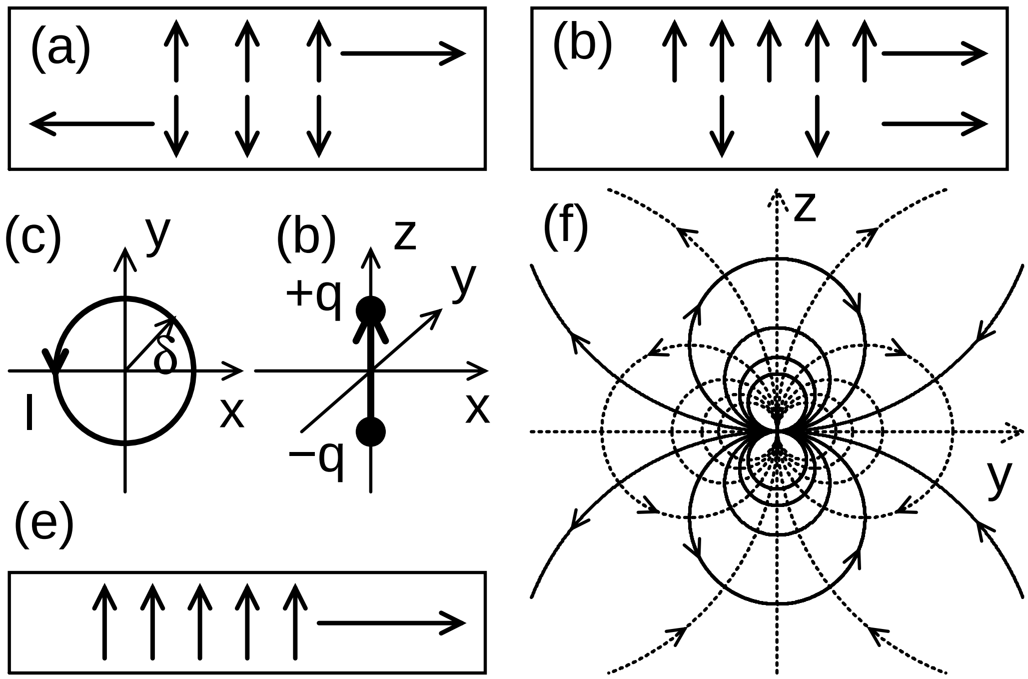

Fig.1

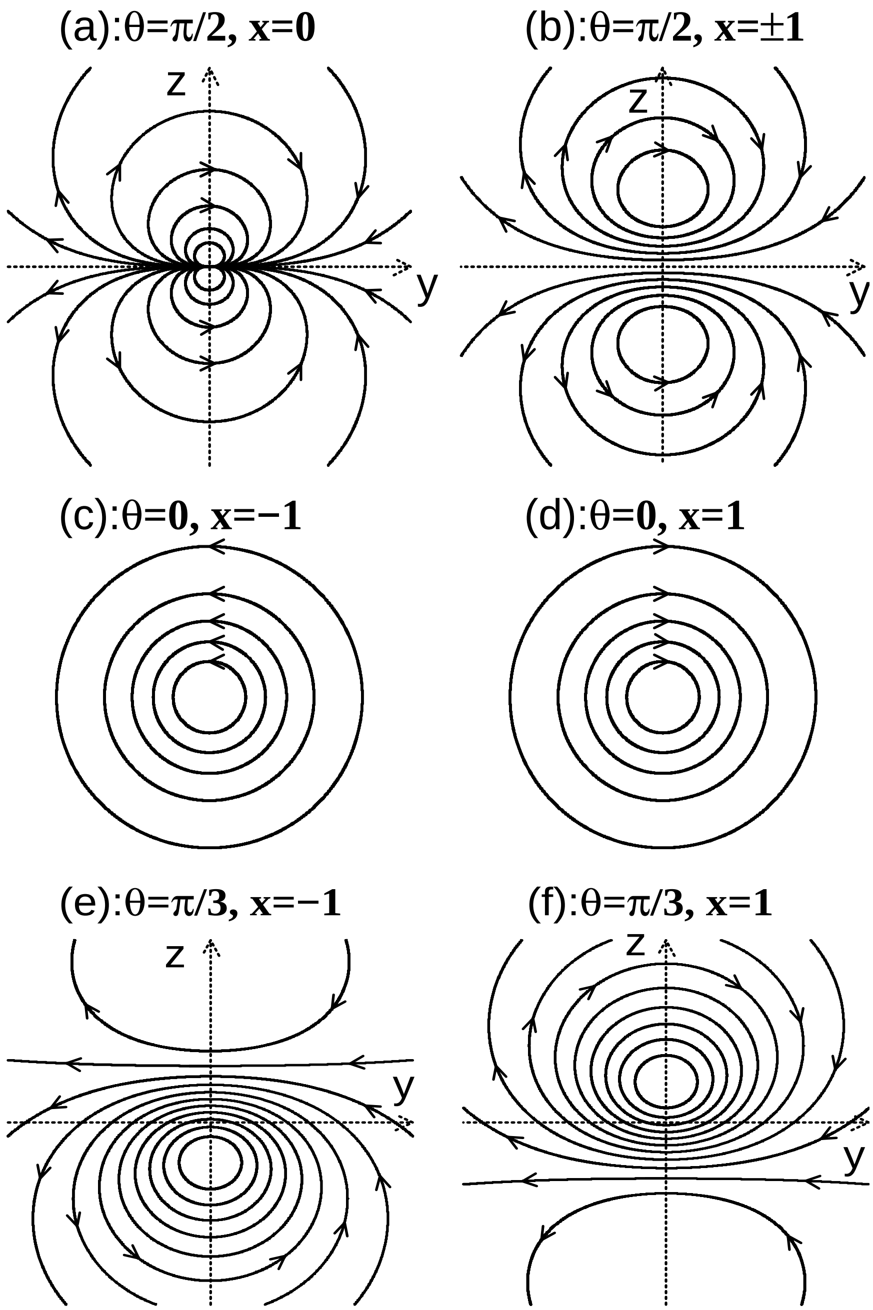

Fig.2